Using New Iterative Method to Find the Exact Solution for a Class of Stiff Systems of Equations

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Abstract: **In this paper, new iterative method (NIM) is used for the solution of nonlinear and linear stiff equations and systems of equations. First we applied NIM to find the solution of stiff equations in series form and then easily converted it into exact solution. The method is explained with some examples. The results show the efficiency and convenience of the NIM which we have used for the solution of stiff system of problems.**

Keywords: **New iterative method; class of stiff systems; scientific work place.**

I. INTRODUCTION

A stiff equation is a differential equation whose solution is difficult to find without using the very small step size. There is no precise definition of the stiffness, but the main idea is that the equation contains some terms that can took to the rapid variation in results.

Consider initial value problem

$$\frac{dy}{dx} = f\big(x, (x)\big)\,, y(x_0) = y_0$$

On the interval $I = [x_0, x_N]$

The stiff differential equations occur in every field of life [1-4] especially in the study of chemical reactions, Reaction-Diffusion-Systems. Some other fields in which stiffness occurs is in electrical circuits, metrology, mechanics, vibrations and oceanography etc.

During last few decades, lot of work has been made on the development of easy efficient and effective methods for stiff problems [1, 2]. The stiff equation's numerical solution for solving nonlinear and linear stiff systems can be found in [3-5].

Recently, the new iterative method developed by Daftardar-Gejji and Hossein Jafri, has acquired a special status. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes. The method has been extensively used in [6–11] among many others. An efficient modification to DJM for solving linear and nonlinear Klein-Gordon equations can be found in [11].

II. THE NEW ITERATIVE METHOD

Consider the following general functional equation

$$y(\bar{x}) = f(\bar{x}) + N\big(y(\bar{x})\big), \qquad (1)$$

Where N is nonlinear from a Banach space B→B, f is a known function and $\bar{x} = (x_1, x_2, \dots\dots, x_n)$. We are looking for a solution y of eq. (1) having the series form

$$y(\bar{x}) = \sum_{n=0}^{\infty} y_i\,(\bar{x}) \qquad (2)$$

The nonlinear operator N can be decomposed as

$$N(\sum_{n=0}^{\infty} y_i) = N(y_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} y_j) - N(\sum_{j=0}^{i-1} y_j)\} \qquad (3)$$

From Eqs. (2) And (3), Eq. (1) is equivalent to

$$\sum_{i=0}^{\infty} y_i = f + N(y_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} y_j) - N(\sum_{j=0}^{i-1} y_j)\}. \qquad (4)$$

We define the recurrence relation

$$y_0 = f,$$
$$y_1 = N(y_0), \qquad (5)$$
$$y_{m+1} = N(y_0 + \cdots + y_m) - N(y_0 + \cdots + y_{m-1})$$

Then

$$(y_1 + \cdots + y_{m+1}) = N(y_0 + \cdots + y_m),$$

And

$$\sum_{i=0}^{\infty} y_i = f + N(\sum_{i=0}^{\infty} y_i). \qquad (6)$$

The k-term approximation solution of Eq. (1) is given by $y = y_0 + y_1 + \cdots + y_{k-1}$

If N contracts i.e.$\parallel N(x) - N(y) \parallel \leq k \parallel x - y \parallel$,$0 < k < 1$,then

$$\parallel y_{m+1} \parallel = \parallel N(y_0 + \cdots + ys_m \parallel - N \parallel y_0 + \cdots + y_{m-1} \parallel \leq k \parallel y_m \parallel \leq k^m \parallel y_0 \parallel, m = 0, 1, 2 \ldots \ldots$$

And series $\sum_{i=0}^{\infty} y_i$uniformly and absolutely converges to solution of equation (1). A unique solution, with respect to Banach fixed point theorem [12].

III. NUMERICAL PROBLEMS OF STIFF SYSTEM

Problem .1 Consider the linear stiff system

$$y_1' = -y_1 - 15y_2 + 15e^{-x}$$
$$y_2' = 15y_1 - y_2 - 15e^{-x}$$

And the initial conditions are $y_1(0) = y_2(0) = 1$,

The corresponding integral equations are as follows

$$y_1(x) = 16 - 15e^{-x} - \int_0^x (y_1 + 15y_2)dx$$

$$y_2(x) = -14 + 15e^{-x} + \int_0^x (15y_1 - y_2)dx$$

Setting $y_{10} = 16 - 15e^{-x}, y_{20} = -14 + 15e^{-x}$ and

$$N_1(y_{10}, y_{20}) = -\int_0^x (y_1 + 15y_2)dx, \qquad N_2(y_{10}, y_{20}) = \int_0^x (15y_1 - y_2)dx$$

Following the algorithm of NIM we obtain following approximations:

$$y_{11} = N_1(y_{10}, y_{20}) = 194x + 210\,e^{-x} - 210$$

$$y_{21} = N_2(y_{10}, y_{20}) = 254x + 240e^{-x} - 240$$

$$y_{12} = N_1(y_{10} + y_{11}, y_{20} + y_{21}) - N_1(y_{10}, y_{20}) = 3810x + 3810e^{-x} - 2002x^2 - 3810$$

$$y_{22} = N_2(y_{10} + y_{11}, y_{20} + y_{21}) - N_2(y_{10}, y_{20}) = 1328x^2 - 2910e^{-x} - 2910x + 2910x_{13}$$

$$y_{13} = N_1(y_{10}+y_{11}+y_{12}, y_{20}+y_{21}+y_{22}) - N_1(y_{10}+y_{11}, y_{20}+y_{21})$$
$$= 19920x^2 - 39840e^{-x} - 39840x - \left(\frac{17918}{3}\right)x^3 + 39840$$

$$y_{23} = N_2(y_{10}+y_{11}+y_{12}, y_{20}+y_{21}+y_{22}) - N_2(y_{10}+y_{11}, y_{20}+y_{21})$$
$$= 30030x^2 - 60060e^{-x} - 60060x - \left(\frac{31358}{3}\right)x^3 + 60060$$

$$y_{14} = N_1(y_{10}+y_{11}+y_{12}+y_{13}, y_{20}+y_{21}+y_{22}+y_{23}) - N_1(y_{10}+y_{11}+y_{12}, y_{20}+y_{21}+y_{22})$$
$$= 470370x^2 - 940740e^{-x} - 940740x - 156790\,x^3 + \left(\frac{122072}{3}\right)x^4 + 940740$$

$$y_{24} = N_2(y_{10}+y_{11}+y_{12}+y_{13}, y_{20}+y_{21}+y_{22}+y_{23}) - N_2(y_{10}+y_{11}+y_{12}, y_{20}+y_{21}+y_{22})$$
$$= 537540x + 537540e^{-x} - 268770x^2 + 89590\,x^3 - \left(\frac{59353}{3}\right)x^4 - 537540$$

The series form of the solution is

$$y_1(x) = 1 - x + \left(\frac{1}{2}\right)x^2 - \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{24}\right)x^4 + \cdots$$

$$y_2(x) = 1 - x + \left(\frac{1}{2}\right)x^2 - \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{24}\right)x^4 + \cdots$$

The exponential form is $y_1(x) = e^{-x}, y_2(x) = e^{-x}$ this is the exact solution.

Problem 2. Consider the nonlinear stiff system

$$y_1' = -1002y_1 + 1000y_2^2$$

$$y_2' = y_1 - y_2 - y_2^2$$

And the initial conditions are $y_1(0) = y_2(0) = 1$

The corresponding integral equations are as follows

$$y_1(x) = 1 + \int_0^x (-1002y_1 + 1000y_2^2)dx$$

$$y_2(x) = 1 + \int_0^x (y_1 - y_2 - y_2^2)dx$$

Setting $y_{10} = 1, y_{20} = 1$ and

$$N_1(y_{10}, y_{20}) = \int_0^x (-1002y_1 + 1000y_2^2)dx$$

$$N_2(y_{10}, y_{20}) = \int_0^x (y_1 - y_2 - y_2^2)dx$$

Following the algorithm of NIM we obtain following approximations:

$$y_{11} = N_1(y_{10}, y_{20}) = -2x$$

$$y_{21} = N_2(y_{10}, y_{20}) = -x$$

$$y_{12} = N_1(y_{10}+y_{11}, y_{20}+y_{21}) - N_1(y_{10}, y_{20}) = \left(\frac{1000}{3}\right)x^3 + 2x^2$$

$$y_{22} = N_2(y_{10}+y_{11}, y_{20}+y_{21}) - N_2(y_{10}, y_{20}) = \left(\frac{1}{2}\right)x^2 - \left(\frac{1}{3}\right)x^3$$

$$y_{13} = N_1(y_{10}+y_{11}+y_{12}, y_{20}+y_{21}+y_{22}) - N_1(y_{10}+y_{11}, y_{20}+y_{21})$$
$$= \left(\frac{1000}{63}\right)x^7 - \left(\frac{500}{9}\right)x^6 + \left(\frac{550}{3}\right)x^5 - \left(\frac{251750}{3}\right)x^4 - \left(\frac{1004}{3}\right)x^3$$

$$y_{23} = N_2(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}) - N_2(y_{10} + y_{11}, y_{20} + y_{21})$$
$$= -(\frac{1}{63})x^7 + (\frac{1}{18})x^6 - (\frac{11}{60})x^5 + \left(\frac{503}{6}\right)x^4 + \left(\frac{1}{6}\right)x^3$$

$$y_{14} = N_1(y_{10} + y_{11} + y_{12} + y_{13}, y_{20} + y_{21} + y_{22} + y_{23}) - N_1(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22})$$
$$= \left(\frac{200}{11907}\right)x^{15} - (\frac{500}{3969})x^{14} + \left(\frac{5050}{7371}\right)x^{13} - (\frac{84475}{378})x^{12} + (\frac{3535705}{4158})x^{11} - \left(\frac{5816\frac{15}{189}}{}\right)x^{10}$$
$$+ \left(\frac{147591700}{189}\right)x^9 - \left(\frac{1391525}{252}\right)x^8 + \left(\frac{1259050}{63}\right)x^7 - \left(\frac{527350}{9}\right)x^6 + \left(\frac{50551100}{3}\right)x^5$$
$$+ \left(\frac{251752}{3}\right)x^4$$

$$y_{24} = N_2(y_{10} + y_{11} + y_{12} + y_{13}, y_{20} + y_{21} + y_{22} + y_{23}) - N_2(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}) = -\left(\frac{1}{59535}\right)x^{15} + \left(\frac{1}{7938}\right)x^{14} - \left(\frac{101}{147420}\right)x^{13} + \left(\frac{3379}{15120}\right)x^{12} - \left(\frac{707141}{831600}\right)x^{11} + \left(\frac{116323}{37800}\right)x^{10} - (\frac{1475917}{1890})x^9 + (\frac{18547}{3360})x^8 - (\frac{25171}{1260})x^7 + (\frac{21083}{360})x^6 - (\frac{505007}{30})x^5 - (\frac{2011}{24})x^4$$

The series form of the solution is

$$y_1(x) = 1 - 2x + 2x^2 - \left(\frac{4}{3}\right)x^3 + \left(\frac{2}{3}\right)x^4 + \cdots$$

$$y_2(x) = 1 - x + \left(\frac{1}{2}\right)x^2 - \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{24}\right)x^4 + \cdots$$

The exponential form is $y_1(x) = e^{-2x}$, $y_2(x) = e^{-x}$ this is exact solution.

Problem. 3 Consider the system of initial value problems

$$y_1' = -y_1$$
$$y_2' = -10y_2$$
$$y_3' = -100y_3$$
$$y_4' = -1000y_4$$

The initial conditions are $y_1(0) = y_2(0) = y_3(0) = y_4(0) = 1,$

The corresponding integral equations are as follows

$$y_1(x) = 1 - \int_0^x y_1 dx, \quad y_2(x) = 1 - 10\int_0^x y_2 dx$$

$$y_3(x) = 1 - 100\int_0^x y_3 dx, \quad y_4(x) = 1 - 1000\int_0^x y_4 dx$$

Setting $y_{10} = y_{20} = y_{30} = y_{40} = 1$ and

$$N_1(y_{10}, y_{20}, y_{30}, y_{40}) = -\int_0^x y_1 dx$$

$$N_2(y_{10}, y_{20}, y_{30}, y_{40}) = -10\int_0^x y_2 dx$$

$$N_3(y_{10}, y_{20}, y_{30}, y_{40}) = -100\int_0^x y_3 dx$$

$$N_4(y_{10}, y_{20}, y_{30}, y_{40}) = -1000\int_0^x y_4 dx$$

Following the algorithm of NIM we obtain following approximations:

$$y_{11} = N_1(y_{10}, y_{20}, y_{30}, y_{40}) = -x$$

$$y_{21} = N_2(y_{10}, y_{20}, y_{30}, y_{40}) = -10x$$

$$y_{31} = N_3(y_{10}, y_{20}, y_{30}, y_{40}) = -100x$$

$$y_{41} = N_4(y_{10}, y_{20}, y_{30}, y_{40}) = -1000x$$

$$y_{12} = N_1(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) - N_1(y_{10}, y_{20}, y_{30}, y_{40}) = \left(\frac{1}{2}\right)x^2$$

$$y_{22} = N_2(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) - N_2(y_{10}, y_{20}, y_{30}, y_{40}) = 50x^2$$

$$y_{32} = N_3(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) - N_3(y_{10}, y_{20}, y_{30}, y_{40}) = 5000x^2$$

$$y_{42} = N_4(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) - N_4(y_{10}, y_{20}, y_{30}, y_{40}) = 500000x^2$$

$$y_{13} = N_1(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) - N_1(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) = -\left(\frac{1}{6}\right)x^3$$

$$y_{23} = N_2(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) - N_2(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) = -\left(\frac{500}{3}\right)x^3$$

$$y_{33} = N_3(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) - N_3(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) = -\left(\frac{500000}{3}\right)x^3$$

$$y_{43} = N_4(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) - N_4(y_{10} + y_{11}, y_{20} + y_{21}, y_{30} + y_{31}, y_{40} + y_{41}) = -\left(\frac{500000000}{3}\right)x^3$$

$$y_{14} = N_1(y_{10} + y_{11} + y_{12} + y_{13}, y_{20} + y_{21} + y_{22} + y_{23}, y_{30} + y_{31} + y_{32} + y_{33}, y_{40} + y_{41} + y_{42} + y_{43}) - N_1(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) = \left(\frac{1}{24}\right)x^4$$

$$y_{24} = N_2(y_{10} + y_{11} + y_{12} + y_{13}, y_{20} + y_{21} + y_{22} + y_{23}, y_{30} + y_{31} + y_{32} + y_{33}, y_{40} + y_{41} + y_{42} + y_{43}) - N_2(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) = \left(\frac{1250}{3}\right)x^4$$

$$y_{34} = N_3(y_{10} + y_{11} + y_{12} + y_{13}, y_{20} + y_{21} + y_{22} + y_{23}, y_{30} + y_{31} + y_{32} + y_{33}, y_{40} + y_{41} + y_{42} + y_{43}) - N_3(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) = \left(\frac{12500000}{3}\right)x^4$$

$$y_{44} = N_4(y_{10} + y_{11} + y_{12} + y_{13}, y_{20} + y_{21} + y_{22} + y_{23}, y_{30} + y_{31} + y_{32} + y_{33}, y_{40} + y_{41} + y_{42} + y_{43}) - N_4(y_{10} + y_{11} + y_{12}, y_{20} + y_{21} + y_{22}, y_{30} + y_{31} + y_{32}, y_{40} + y_{41} + y_{42}) = \left(\frac{125\ 000000000}{3}\right)x^4$$

The series form of the solution is

$$y_1(x) = 1 - x + \left(\frac{1}{2}\right)x^2 - \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{24}\right)x^4 + \cdots$$

$$y_2(x) = 1 - 10x + 50x^2 - \left(\frac{500}{3}\right)x^3 + \left(\frac{1250}{3}\right)x^4 + \cdots$$

$$y_3(x) = 1 - 100x + 5000x^2 - \left(\frac{500000}{3}\right)x^3 + \left(\frac{12500000}{3}\right)x^4 + \cdots$$

$$y_4(x) = 1 - 1000x + 500000x^2 - \left(\frac{500000000}{3}\right)x^3 + \left(\frac{125\ 000000000}{3}\right)x^4 + \cdots$$

The exponential form of the equations is $y_1(x) = e^{-x}, y_2(x) = e^{-10x}, y_3(x) = e^{-100x}, y_4(x) = e^{-1000x}$ this is the exact solution.

IV. CONCLUSION

The purpose of the study is to show the efficiency of the new iterative method in obtaining the exact solution of the stiff systems of equations. The solution are obtained in easy and direct way without taking any assumptions that may change the problem's behavior. These results show that NIM reduces the calculations size and gives the rapid solution if we compare it with other iterative methods. Using few approximations we achieve the high accuracy.

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